ANALYSIS OF NON-THIN SHELLS

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A method of analyzing non-thin shells of constant thickness $2h$ is proposed, which is based on some properties of Legendre polynomials.

1. For the case when the middle surface of a shell is closed it is proved that the three-dimensional problem of elasticity theory on the construction of the stress-strain state of such a shell separates into two problems. Let the shell be referred to the curvilinear coordinates θ^1, θ^2 (it is assumed throughout that the Greek indices take on the values 1, 2, and the Latin indices, the values 1, 2, 3).

Then the first of the mentioned problems will be to construct the stress-strain state in which the displacements vary linearly with θ^3. It will be called the linear-thickness problem.

The second problem is to construct the stress-strain state characterized by the fact that the resultant force and resultant moment would be zero therein in any element normal to the middle surface. This requirement is analogous to conditions characterizing the so-called boundary layer. Hence, we call the second the problem of the pseudo-boundary layer. The solution of both problems should individually be an exact solution of the three-dimensional elasticity theory equations, and together they should yield the solution of the original problem.

In the shell with open middle surface conditions on the side surfaces will not be satisfied. To eliminate these discrepancies it is also necessary to utilize the boundary layers.

Series expansions of the stresses, strains, and displacements in orthogonal Legendre polynomials in the form

$$\sigma^{ij} = \sum_{n=0}^{\infty} \sigma_{(n)}^{ij} P_n\left(\frac{\theta^3}{h}\right), \qquad \gamma_{ij} = \sum_{n=0}^{\infty} \gamma_{ij}^{(n)} P_n\left(\frac{\theta^3}{h}\right), \qquad u_i = \sum_{n=0}^{\infty} u_i^{(n)} P_n\left(\frac{\theta^3}{h}\right) \quad (1.1)$$

are utilized below.

Here $P_n\ (\theta^3/h)$ is the Legendre polynolial of order n.

2. Let us designate the linear-thickness problem of elasticity theory to be that whose solution possesses the following properties:

a) Satisfies the strain-displacement relationships, as well as Hooke's law for three-dimensional elasticity theory;

b) Corresponds to boundary stresses on the surfaces S_+ and S_-, and to mass forces whose true distributions can be replaced by any other distributions under the condition that the corresponding resultant force and resultant moment in any normal element of the shell remain invariant (we call these distributions of the boundary stresses and mass forces the equivalent auxiliary distributions);

c) Contain sufficient arbitrariness to satisfy the conditions on the normal edge surface $\theta^1 = ct$ for the following stress or displacement components in the Legendre expansions

$$\sigma_{(0)}^{11},\ \sigma_1^{11},\ \sigma_{(0)}^{12},\ \sigma_{(1)}^{12},\ \sigma_{(0)}^{13},\ u_1^{(0)},\ u_1^{(1)},\ u_2^{(0)},\ u_2^{(1)},\ u_3^{(0)}$$

Utilizing the notation from [1], let us consider components of the elastic displacement vector in the form

$$u_\alpha = u_\alpha^{(0)} + x u_\alpha^{(1)}, \qquad u_3 = u_3^{(0)} \tag{2.1}$$

where $u_\alpha^{(0)}$, $u_\alpha^{(1)}$, $u_3^{(0)}$ are functions of θ^α and

$$x = \theta^3 / h \tag{2.2}$$

Proceeding from the representations (2.1), the strain components can be written as finite Legendre expansions

$$\gamma_{ij} = \sum_{n=0}^{N} \gamma_{ij}^{(n)} P_n(x) \tag{2.3}$$

The coefficients of these expansions are

$$2\gamma_{\alpha\beta}^{(0)} = u_\alpha^{(0)}|_\beta + u_\beta^{(0)}|_\alpha - 2b_{\alpha\beta}u_3^{(0)} - {}^1/_3\, h\,(b_\beta{}^\lambda u_\lambda{}^{(1)}|_\alpha + b_\alpha{}^\lambda u_\lambda{}^{(1)}|_\beta) \tag{2.4}$$

$$2\gamma_{\alpha\beta}^{(1)} = u_\alpha{}^{(1)}|_\beta + u_\beta{}^{(1)}|_\alpha - h\,(b_\beta{}^\lambda u_\lambda{}^{(0)}|_\alpha + b_\alpha{}^\lambda u_\lambda{}^{(0)}|_\beta) + h\,(b_\beta{}^\lambda b_{\lambda\alpha} + b_\alpha{}^\lambda b_{\lambda\beta})\, u_3{}^{(0)}$$

$$2\gamma_{\alpha\beta}^{(2)} = -{}^2/_3\, h\,(b_\beta{}^\lambda u_\lambda{}^{(1)}|_\alpha + b_\alpha{}^\lambda u_\lambda{}^{(1)}|_\beta)$$

$$2\gamma_{\alpha\beta}^{(n)} = 0 \quad (n > 2), \qquad 2\gamma_{\alpha 3}^{(0)} = \frac{\partial u_3{}^{(0)}}{\partial \theta^\alpha} + b_\alpha{}^\lambda u_\lambda{}^{(0)} + \frac{1}{h} u_\alpha{}^{(1)}$$

$$2\gamma_{\alpha 3}^{(n)} = 0 \quad (n > 0), \qquad 2\gamma_{33}^{(n)} = 0 \quad (n = 0, 1, 2, \ldots)$$

The customary rule for summation over repeated subscripts is used, and the vertical bars correspond to covariant differentiation in the metric of the middle surface.

Utilizing the fact that $\gamma_{33} = 0$ because of (2.4), let us write the physical relationships of elasticity theory thus:

$$\sigma^{\omega\varphi} = E^{\omega\varphi\alpha\beta}\gamma_{\alpha\beta}, \qquad \sigma^{\varphi 3} = E^{\varphi 3\alpha 3}\gamma_{\alpha 3} \tag{2.5}$$

Here

$$E^{\omega\varphi\alpha\beta} = \mu\,(\delta_\lambda{}^\varphi - xhb_\lambda{}^\varphi)\left(\frac{g}{a}\right)^{1/2} \frac{1}{g^2}\left(G^{\omega\alpha}G^{\lambda\beta} + G^{\omega\beta}G^{\lambda\alpha} + \frac{2\eta}{1-2\eta}\, G^{\omega\lambda}G^{\alpha\beta}\right)$$

$$E^{\varphi 3\alpha 3} = 2\mu \left(\frac{g}{a}\right)^{1/2} \frac{1}{g^2}\, G^{\varphi\alpha}G^{33}$$

The cofactors of g_{ij} in the determinant $g = |g_{ij}|$ are denoted by G^{ij} in the last equalities. They will be polynomials in x.

Let us introduce the notation

$$\frac{2}{2m+1} E_{(n),(m)}^{\omega\varphi\alpha\beta} = \int_{-1}^{+1} P_m(x) P_n(x) E^{\omega\varphi\alpha\beta}\, dx$$

$$\frac{2}{2m+1} E_{(n),(m)}^{\omega 3\alpha 3} = \int_{-1}^{+1} P_m(x) P_n(x) E^{\omega 3\alpha 3} dx \tag{2.6}$$

where $E_{(n),(m)}^{\omega\varphi\alpha\beta}$, $E_{(n),(m)}^{\omega 3\alpha 3}$ are coefficients of the Legendre function expansions $P_n(x)E^{\omega\varphi\alpha\beta}$, $P_n(x)E^{\omega 3\alpha 3}$, respectively, and we note that the integrals (2.6) are computed directly, wherein only rational functions of x enter in whose denominator the following expression is contained

$$(g/a)^{3/2} = (1 - hxe_\lambda^\lambda + h^2x^2K)^3$$

Therefore

$$P_n(x)\, E^{\omega\varphi\alpha\beta} = \sum_{m=0}^{\infty} E_{(n),(m)}^{\omega\varphi\alpha\beta} P_m(x), \qquad P_n(x)\, E^{\omega 3\alpha 3} = \sum_{m=0}^{\infty} E_{(n),(m)}^{\omega 3\alpha 3} P_m(x) \tag{2.7}$$

By using these formulas the coefficients of the Legendre expansions for the stresses can be determined

$$\sigma^{\omega\varphi} = \sum_{n=0}^{\infty} \sigma_{(n)}^{\omega\varphi} P_n(x), \qquad \sigma^{\varphi 3} = \sum_{n=0}^{\infty} \sigma_{(n)}^{\varphi 3} P_n(x) \tag{2.8}$$

since according to (2.6), (2.7)

$$\sigma_{(n)}^{\omega\varphi} = \sum_{q=0}^{2} E_{(q),(n)}^{\omega\varphi\alpha\beta}\gamma_{\alpha\beta}^{(q)}, \qquad \sigma_{(n)}^{\varphi 3} = E_{(0),(n)}^{\varphi 3\alpha 3}\gamma_{\alpha 3}^{(0)} \tag{2.9}$$

The equilibrium equations of three-dimensional elasticity theory are written thus:

$$\sigma^{\alpha\beta}|_\alpha - b_\alpha{}^\beta\sigma^{\alpha 3} + \frac{\partial\sigma^{3\beta}}{\partial\theta^3} + F^\beta = 0, \ \sigma^{\alpha 3}|_\alpha + b_{\alpha\beta}\sigma^{\alpha\beta} + \frac{\partial\sigma^{33}}{\partial\theta^3} + F^3 = 0 \ . \tag{2.10}$$

It follows from the orthogonality condition for Legendre polynomials that

$$\int_{-1}^{+1} P_n(x)\,dx = 0, \qquad \int_{-1}^{1} xP_n(x)\,dx = 0 \qquad (n \geqslant 2)$$

Hence, members with subscript n greater than one in the expansions (2.8) will correspond to some stress distributions self-equilibrated through the thickness. These members may be discarded in the solution of the linear problem.

We then obtain from (2.8) and (2.10)

$$2\left(\sigma_{(0)}^{\alpha\beta}|_\alpha - b_\alpha{}^\beta\sigma_{(0)}^{\alpha 3}\right) + \left(\frac{\sigma^{3\beta}(1) - \sigma^{3\beta}(-1)}{h} + \int_{-1}^{+1} P_0 F^\beta dx\right) = 0$$

$$2\left(\sigma_{(0)}^{\alpha 3}|_\alpha + b_{\alpha\beta}\sigma_{(0)}^{\alpha\beta}\right) + \left(\frac{\sigma^{33}(1) - \sigma^{33}(-1)}{h} + \int_{-1}^{+1} P_0 F^3 dx\right) = 0$$

$$\frac{2}{3}\left(\sigma_{(1)}^{\alpha\beta}|_\alpha - \frac{3}{h}\sigma_{(0)}^{\beta 3}\right) + \left(\frac{\sigma^{3\beta}(1) + \sigma^{3\beta}(-1)}{h} + \int_{-1}^{+1} P_1(x) F^\beta dx\right) = 0 \tag{2.11}$$

On the basis of the properties of the solution of the linear-thickness problem, we note that the components R^β, R^3 and C^β of the resultant force and the resultant moment are

$$R^\beta = h\left(\frac{\sigma^{3\beta}(1) - \sigma^{3\beta}(-1)}{h} + \int_{-1}^{+1} F^\beta dx\right)$$

$$R^3 = h\left(\frac{\sigma^{33}(1) - \sigma^{33}(-1)}{h} + \int_{-1}^{+1} F^3 dx\right)$$

$$C^\beta = h^2\left(\frac{\sigma^{3\beta}(1) + \sigma^{3\beta}(-1)}{h} + \int_{-1}^{+1} xF^\beta dx\right)$$

It results from these relationships that the first coefficients of the Legendre expansions for the mass forces are expressed by Formulas

$$F_{(0)}^\beta = \frac{1}{2h}[R^\beta - (\sigma^{3\beta}(1) - \sigma^{3\beta}(-1))], \qquad F_{(0)}{}^3 = \frac{1}{2h}[R^3 - (\sigma^{33}(1) - \sigma^{33}(-1))]$$

$$F_{(1)}^\beta = \frac{3}{2h^2}[C^\beta - h(\sigma^{3\beta}(1) + \sigma^{3\beta}(-1))] \tag{2.12}$$

in which the boundary values $\sigma^{3\beta}$, σ^{33} are considered known.

By using these equalities the system of equations (2.11) can be written as

$$\sigma_{(0)}^{\alpha\beta}|_\alpha - b_\alpha{}^\beta\sigma_{(0)}^{\alpha 3} + \frac{1}{2h}R^\beta = 0 \tag{2.13}$$

$$\sigma_{(0)}^{\alpha 3}|_\alpha + b_{\alpha\beta}\sigma_{(0)}^{\alpha\beta} + \frac{1}{2h}R^3 = 0, \qquad \sigma_{(1)}^{\alpha\beta}|_\alpha - \frac{3}{h}\sigma_{(0)}^{\beta 3} + \frac{3}{2h^2}C^\beta = 0$$

These equalities are a system of equilibrium equations for the linear-thickness problem. The corresponding physical equations are obtained from (2.9)

$$\sigma_{(0)}^{\alpha\beta} = \sum_{q=0}^{2} E_{(q).(0)}^{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}^{(q)}, \qquad \sigma_{(1)}^{\alpha\beta} = \sum_{q=0}^{2} E_{(q).(1)}^{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}^{(q)}, \qquad \sigma_{(0)}^{\alpha 3} = E_{(0).(0)}^{\alpha 3\lambda 3}\gamma_{\lambda 3}^{(0)}$$

The strains are here determined by the geometric relationships (2.4) of the linear-thickness problem. Expressing the stresses in terms of the displacements in the equilibrium equations (2.13), we obtain a system of five equations with the five unknowns $u_\alpha^{(0)}, u_\alpha^{(1)}, u_3^{(0)}$.

The three-dimensional components u_α, u_3 of the displacements are determined from (2.1) in terms of the unknowns of this system.

The stresses σ^{ij} are determined from the geometric and physical equations of three-dimensional elasticity theory, and we obtain the final expressions for the mass forces from the equilibrium equations

$$F_{(l)}^{\beta} = -\sigma^{\alpha\beta}|_{\alpha} + b_{\alpha}^{\beta}\sigma^{\alpha 3} - \frac{\partial\sigma^{3\beta}}{\partial\theta^3}, \quad F_{(l)}^{3} = -\sigma^{\alpha 3}|_{\alpha} - b_{\alpha\beta}\sigma^{\alpha\beta} - \frac{\partial\sigma^{33}}{\partial\theta^3}$$

which satisfy the integral conditions of the linear-thickness problem according to (2.12). The derived system of equations of the linear-thickness problem permit compliance with the five boundary conditions corresponding to the property (c) of the linear-thickness problem.

Since the displacements are known, we indeed obtain the boundary stresses corresponding to the linear-thickness problem $\sigma_{(l)}^{3i}(\theta^\lambda, \pm 1)$, which generally do not evidently agree with the true values. The solution of the pseudo-boundary-layer problem also permits correction of these boundary results.

It can be shown that if stress-strain states with index of variability greater than $^1/_2$ are not taken into account, and terms retained to satisfy the fifth boundary condition are discarded, then a modification of classical theory with error h/R as compared to one will be obtained.

3. The pseudo-boundary-layer problems should satisfy the following conditions:

a) Stresses corresponding to the corrections should satisfy the conditions

$$\sigma^{3i}(\theta^\lambda, \pm 1) - \sigma_{(l)}^{3i}(\theta^\lambda, \pm 1) = \sigma_{(\lambda)}^{3i}(\theta^\lambda, \pm 1)$$

on S_+ and S_- ;

b) Internal stresses corresponding to any cross-sectional element normal to the middle surface should have zero resultant force and resultant moment from which it follows that $\sigma_{(0)}^{\alpha i}$ and $\sigma_{(1)}^{\alpha\beta}$ equal zero;

c) The requirement $u_i^{(0)} = 0$, $u_\alpha^{(1)} = 0$ should similarly be satisfied in the Legendre expansions for the displacements;

d) Mass forces corresponding to the correction pseudo-boundary-layer problem are the difference between the true mass forces and the mass forces $F_{(l)}^{i}$ obtained from the solution of the linear-thickness problem.

The pseudo-boundary layer problem is not predetermined since compliance with the condition on the edge normal surface is not required.

There results from the properties of the linear-thickness and boundary layer problems that the resultant force and resultant moment of both the surface, and the mass forces should equal zero.

We separate the construction of the solution of the pseudo-boundary-layer problem into two stages.

The first stage is to determine a particular solution of the three-dimensional equations of elasticity theory which will satisfy conditions (a), (b), (c) in Section 3.

The second stage is also to solve the unpredetermined problem. Homogeneous boundary conditions are posed on S_+ and S_- for it so as not to violate the condition (d) satisfied in the first stage. The mass forces (statically equivalent to zero) should equal the difference between the actual mass forces and the mass forces obtained in solving the linear-thickness problem and in the first stage of solving the pseudo-boundary-layer problem.

Utilizing the fact that satisfaction of the conditions on the edge normal surface is not certain in the pseudo-boundary-layer problem, we assume the middle surface to be referred to the lines of curvature.

The coordinate system used will then be tri-orthogonal.

3.1. First stage in solving the pseudo-boundary-layer problem. Let us examine the boundary conditions on S_+ and S_- by formulating them as:

$$\tau_{13}(\pm h) = \frac{H_1^-(\pm h)\, H_2^-(\pm h)\, t_1^{\pm}(\theta^1, \theta^2)}{H} \qquad (1 \leftrightarrow 2)$$

$$\tau_{33}(\pm h) = \frac{H_1^-(\pm h)\, H_2^-(\pm h)\, s^{\pm}(\theta^1, \theta^2)}{H}$$

Here $\tau_{13}(\pm h)$, $\tau_{23}(\pm h)$, $\tau_{33}(\pm h)$ are known functions representing the boundary correction stresses.

Let us introduce the notation

$$H_1(\theta^3) = A_1\left(1 + \frac{\theta^3}{R_1}\right), \qquad H_1^-(\theta^3) = A_1\left(1 - \frac{\theta^3}{R_1}\right) \qquad (1 \leftrightarrow 2)$$

$$H = A_1 A_2 (1 - h^2/R_1^2)^{1/2}\, (1 - h^2/R_2^2)^{1/2}$$

For the sequel, let us note that any boundary conditions can be represented as the superposition of boundary conditions corresponding to

$$t_1^{\pm}(\theta^1, \theta^2) = t_1(\theta^1, \theta^2), \quad s^{\pm}(\theta^1, \theta^2) = \pm s(\theta^1, \theta^2) \qquad (3.1)$$

$$t_1^{\pm}(\theta^1, \theta^2) = \pm t_1(\theta^1, \theta^2), \quad s^{\pm}(\theta^1, \theta^2) = s(\theta^1, \theta^2) \qquad (3.2)$$

The requirement formulated earlier that the stresses be statically equivalent on any section normal to the middle surface will be assured if the following conditions are imposed: $T_1 = 0, \quad S_{12} = 0, \quad N_1 = 0, \quad G_1 = 0, \quad H_{12} = 0 \quad (1 \leftrightarrow 2)$

Stress resultants and moments are on the left here. Expressing them in terms of the internal stresses of a three-dimensional medium occupied by the shell, we will have

$$\int_{-h}^{+h} H_2 \tau_{11}\, d\theta^3 = 0, \quad \int_{-h}^{+h} \theta^3 H_2 \tau_{11}\, d\theta^3 = 0, \quad \int_{-h}^{+h} H_2 \tau_{12}\, d\theta^3 = 0$$

$$\int_{-h}^{+h} \theta^3 H_2 \tau_{12}\, d\theta^3 = 0, \quad \int_{-h}^{+h} H_2 \tau_{13}\, d\theta^3 = 0 \qquad (1 \leftrightarrow 2) \qquad (3.3)$$

Let us note that these conditions will be satisfied if expansions of the following kind are taken for the displacements

$$v_1 = H_1^2H_2^2v_1^*, \qquad v_1^* = \sum_{n=6}^{\infty} V_1^{(n)}(\theta^1, \theta^2)P_n(x) \qquad (1 \leftrightarrow 2) \tag{3.4}$$

$$v_3 = H_1^2H_2^2v_3^*, \qquad v_3^* = \sum_{n=6}^{\infty} V_3^{(n)}(\theta^1, \theta^2)P_n(x)$$

Here $V_1^{(n)}(\theta^1, \theta^2)$, $V_2^{(n)}(\theta^1, \theta^2)$, $V_3^{(n)}(\theta^1, \theta^2)$ are arbitrary functions.

For τ_{13}, τ_{23}, τ_{33} let us take the following expressions which satisfy the boundary conditions:

$$\tau_{13} = \frac{H_1^-H_2^-P_k(x)t_1(\theta^1, \theta^2)}{H} + \frac{1}{H_1}\frac{\partial\Phi(\theta^1, \theta^2, x)}{\partial\theta^1} + \frac{H_1}{h}\frac{\partial\Psi_1(\theta^1, \theta^2, x)}{\partial x} \qquad (1 \leftrightarrow 2) \tag{3.5}$$

$$\tau_{33} = \frac{H_1^-H_2^-P_Q(x)s(\theta^1, \theta^2)}{H} + S(\theta^1, \theta^2, x) \tag{3.6}$$

wherein the functions

$$\frac{\partial\Phi}{\partial\theta^1}, \quad \frac{\partial\Phi}{\partial\theta^2}, \quad \frac{\partial\Psi_1}{\partial x}, \quad \frac{\partial\Psi_2}{\partial x}, \quad S$$

should equal zero on S_+ $(x = 1)$ and S_- $(x = -1)$.

The degree K is even, and Q is odd for conditions of the form (3.1), and, conversely, for conditions of the form (3.2).

From the geometric and physical relationships it follows:

$$\frac{1}{H_1}\frac{\partial v_3}{\partial\theta^1} + \frac{H_1}{h}\frac{\partial}{\partial x}\left(\frac{v_1}{H_1}\right) = \frac{2(1+\nu)}{E}\left[\frac{H_1^-H_2^-P_K(x)t_1(\theta^1, \theta^2)}{H} + \right.$$

$$\left. + \frac{1}{H_1}\frac{\partial\Phi}{\partial\theta^1} + \frac{H_1}{h}\frac{\partial\Psi_1}{\partial x}\right] \qquad (1 \leftrightarrow 2) \tag{3.7}$$

$$\frac{1}{h}\frac{\partial v_3}{\partial x} - \frac{\nu}{1-\nu}\frac{1}{H_1H_2}\left(\frac{\partial H_2v_1}{\partial\theta^1} + \frac{\partial H_1v_2}{\partial\theta^2}\right) + \frac{\nu}{1-\nu}\frac{1}{hH_1H_2}\frac{\partial H_1H_2}{\partial x}v_3 =$$

$$= \frac{(1+\nu)(1-2\nu)}{E(1-\nu)}\left[\frac{H_1^-H_2^-P_Q(x)s(\theta^1, \theta^2)}{H} + S\right] \tag{3.8}$$

Relationship (3.7) is satisfied if

$$v_1 = \frac{2(1+\nu)}{E}H_1\left(\frac{ht_1(\theta^1, \theta^2)}{H}\int_0^x \frac{H_1^-H_2^-P_k(x)}{H_1}dx + \Psi_1(\theta^1, \theta^2, x)\right) \qquad (1 \leftrightarrow 2) \tag{3.9}$$

$$v_3 = \frac{2(1+\nu)}{E}\Phi(\theta^1, \theta^2, x)$$

and the condition

$$\left.\frac{\partial\Phi}{\partial\theta^1}\right|_{x=\pm1} = 0 \qquad (1 \leftrightarrow 2)$$

will be satisfied if

$$v_3^* = \Lambda_1(\theta^1, \theta^2)(P_9(x) - P_7(x)) + \Lambda_2(\theta^1, \theta^2)(P_8(x) - P_6(x))$$

We therefore obtain

$$\Phi = \frac{Ev_3}{2(1+\nu)} = \tag{3.10}$$

$$= \frac{E}{2(1+\nu)}H_1^2H_2^2[\Lambda_1(\theta^1, \theta^2)(P_9(x) - P_7(x)) + \Lambda_2(\theta^1, \theta^2)(P_8(x) - P_6(x))]$$

By using the expansion (3.4) and Formula (3.9) we obtain

$$\Psi_1 = -\frac{ht_1(\theta^1, \theta^2)}{H}\int\limits_0^x \frac{H_1^{-}H_2^{-}P_k(x)}{H_1}\,dx +$$

$$+\frac{E}{2(1+\nu)}H_1H_2^2\sum_{n=6}^{\infty} V_1^{(n)}(\theta^1, \theta^2)\,P_n(x) \qquad (1\leftrightarrow 2) \tag{3.11}$$

The functions Ψ_1 and Ψ_2 should satisfy the above-mentioned conditions

$$\left.\frac{\partial \Psi_1}{\partial x}\right|_{x=\pm 1} = 0 \qquad (1\leftrightarrow 2) \tag{3.12}$$

and τ_{13}, τ_{23} given in the form (3. 5), should satisfy the last conditions of (3. 3).

Performing the appropriate substitution , taking account of (3. 10), and assuming $k > 4$ in (3. 11), we obtain

$$\int\limits_{-1}^{+1} H_1H_2\frac{\partial \Psi_1}{\partial x}\,dx = 0 \qquad (1\leftrightarrow 2) \tag{3.13}$$

According to (3. 11) and (3. 13), we will have

$$\int\limits_{-1}^{+1} H_1H_2\left[-\frac{ht_1(\theta^1, \theta^2)}{H}\frac{H_1^{-}H_2^{-}}{H_1}P_K(x) + \frac{E}{2(1+\nu)}\frac{\partial(H_1H_2^2)}{\partial x}\times\right. \qquad (1\leftrightarrow 2)$$

$$\left.\times\sum_{n=6}^{\infty} V_1^{(n)}(\theta^1, \theta^2)\,P_n(x) + \frac{E}{2(1+\nu)}H_1H_2^2\sum_{n=6}^{\infty} V_1^{(n)}(\theta^1, \theta^2)\,P_n'(x)\right]dx = 0$$

or

$$\int\limits_{-1}^{+1} H_1^2H_2^3\sum_{n=6}^{\infty} V_1^{(n)}(\theta^1, \theta^2)\,P_n'(x)\,dx = 0 \qquad (1\leftrightarrow 2)$$

Integrating by parts we obtain the equivalent conditions

$$H_1^2H_2^3\sum_{n=6}^{\infty} V_1^{(n)}(\theta^1, \theta^2)\,P_n(x)\Big|_{x=-1}^{x=1} = 0 \qquad (1\leftrightarrow 2)$$

They will be satisfied if it is assumed that

$$\sum_{n=6}^{\infty} V_1^{(n)}(\theta^1, \theta^2)\,P_n(x) =$$

$$= C_{v_1}(\theta^1, \theta^2)(P_9(x) - P_7(x)) + D_{v_1}(\theta^1, \theta^2)(P_8(x) - P_6(x)) \qquad (1\leftrightarrow 2)$$

Hence, utilizing (3. 11), we obtain formulas for the functions Ψ_1, Ψ_2

$$\Psi_1 = -\frac{ht_1(\theta^1, \theta^2)}{H}\int\limits_0^x \frac{H_1^{-}H_2^{-}P_K(x)}{H_1}\,dx + \quad (1\leftrightarrow 2)$$

$$+\frac{E}{2(1+\nu)}H_1H_2^2\left[C_{v_1}(\theta^1, \theta^2)(P_9(x) - P_7(x)) + D_{v_1}(\theta^1, \theta^2)(P_8(x) - P_6(x))\right]$$

Conditions (3. 12) permit determination of the functions

$$C_{v_1}(\theta^1, \theta^2),\ D_{v_1}(\theta^1, \theta^2),\ C_{v_2}(\theta^1, \theta^2),\ D_{v_2}(\theta^1, \theta^2)$$

Applying Formulas

$$P_{n+1}'(x) - P_{n-1}'(x) = (2n+1)P_n(x),\ P_n(\pm 1) = (\pm 1)^n$$

we obtain the system

$$17C_{v_1}(\theta^1,\theta^2)+15D_{v_1}(\theta^1,\theta^2)=\frac{2(1+\nu)}{E}\frac{ht_1(\theta^1,\theta^2)}{H}\left(\frac{H_1^- H_2^-}{H_1^2H_2^2}\right)_{x=1}$$

$$(1\leftrightarrow 2)$$

$$17C_{v_1}(\theta^1,\theta^2)-15D_{v_1}(\theta^1,\theta^2)=\frac{2(1+\nu)}{E}\frac{(-1)^K ht_1(\theta^1,\theta^2)}{H}\left(\frac{H_1^- H_2^-}{H_1^2H_2^2}\right)_{x=-1}$$

Therefore, the functions Ψ_1, Ψ_2 are determined.

Formula (3.8) permits finding the functions $S(\theta^1,\theta^2,x)$, and from the conditions

$$S(\theta^1,\theta^2,\pm 1)=0$$

there is obtained a system determining the functions $\Lambda_1(\theta^1,\theta^2)$ and $\Lambda_2(\theta^1,\theta^2)$

$$17\Lambda_1(\theta^1,\theta^2)+15\Lambda_2(\theta^1,\theta^2)=\frac{h(1+\nu)(1-2\nu)}{E(1-\nu)}\frac{s(\theta^1,\theta^2)}{H}\left(\frac{H_1^- H_2^-}{H_1^2H_2^2}\right)_{x=1}$$

$$(Q>0)$$

$$17\Lambda_1(\theta^1,\theta^2)-15\Lambda_2(\theta^1,\theta^2)=\frac{h(1+\nu)(1-2\nu)}{E(1-\nu)}\frac{(-1)^Q s(\theta^1,\theta^2)}{H}\left(\frac{H_1^- H_2^-}{H_1^2H_2^2}\right)_{x=-1}$$

Therefore, the expansions of v_1^*, v_2^*, v_3^* have been determined, and the displacements v_1, v_2, v_3 have been obtained. The stresses τ_{13}, τ_{23}, τ_{33} which will satisfy conditions of the type (3.1) or (3.2) depending on the choice of K and Q in (3.5), (3.6) have also been constructed.

The remaining stresses τ_{11}, τ_{22}, τ_{33} are found by differentiation from the geometrical and physical relationships and we find the mass forces from the equilibrium equations.

Therefore, a particular solution corresponding to the first stage of the pseudo-boundary layer problem has been effectively constructed.

3.2. Second stage of solving the pseudo-boundary layer problem. The solution corresponding to this stage will be constructed for the particular case of a flat plate. Series expansions of the type

$$v=v_{1,0}{}^{\circ}P_0{}^{\circ}+v_{1,1}{}^{\circ}P_1{}^{\circ}+v_{1,1}{}^{1}P_1{}^{1}+v_{1,2}{}^{1}P_2{}^{1}+\sum_{n=2}^{\infty}v_{1,n}{}^{2}P_n{}^{2} \tag{3.14}$$

$$P_K{}^{\circ}=P_K,\qquad P_K{}^{1}=\frac{P_{K+1}{}^{\circ}-P_{K-1}{}^{\circ}}{2K+1},\qquad P_K{}^{2}=\frac{P_{K+1}{}^{1}-P_{K-1}{}^{1}}{2K+1} \tag{3.15}$$

are taken for the displacements assumed continuous for $|x|<1$.

Noting that

$$\frac{dP_K{}^{2}}{dx}=P_K{}^{1},\qquad \frac{dP_K{}^{1}}{dx}=P_K{}^{\circ}=P_K,\qquad P_K{}^{2}(\pm 1)=P_K{}^{1}(\pm 1)=0$$

and substituting (3.15) into (3.14), we obtain

$$v_1=\sum_{n=0}^{\infty}v_{1,n}P_n(x) \tag{3.16}$$

From the equivalence of expansions of the form (3.14), (3.16), and taking account of the functional properties of the Legendre polynolials, we obtain that the series (3.14) and (3.16) have an identical value at each interior point of the interval $-1\leqslant x\leqslant 1$. For $x=\pm 1$ the coefficients $v_{1,0}{}^{\circ}, v_{1,1}{}^{\circ}, v_{1,1}{}^{1}, v_{1,2}{}^{1}$ can be defined so that the series (3.14) would take the value $v_1(\pm 1)$, and its derivative the value $(\partial v_1/\partial x)_{x=\pm 1}$.

Imposing conditions (c) and (d), and also demanding that the stresses be zero on S_+ and

S_-, we obtain $$v_{1,0}{}^0 = v_{1,1}{}^0 = v_{1,1}{}^1 = v_{1,2}{}^1 = v_{1,2}{}^2 = v_{1,3}{}^2 = 0$$

$$v_{2,0}{}^0 = v_{2,1}{}^0 = v_{2,1}{}^1 = v_{2,2}{}^1 = v_{2,2}{}^2 = v_{2,3}{}^2 = 0$$

$$v_{3,0}{}^0 = v_{3,1}{}^0 = v_{3,1}{}^1 = v_{3,2}{}^1 = v_{3,2}{}^2 = 0$$

Hence

$$v_1 = \sum_{n=4}^{\infty} v_{1,n}{}^2 P_n{}^2 \quad (1 \leftrightarrow 2) \qquad v_3 = \sum_{n=3}^{\infty} v_{3,n}{}^2 P_n{}^2 \tag{3.17}$$

The arbitrary functions $v_{1,n}{}^2$, $v_{2,n}{}^2$, $v_{3,n}{}^2$ are determined from the equilibrium equations in displacements

$$\left(\Delta v_1 + \frac{1}{h^2}\frac{\partial^2 v_1}{\partial x^2}\right) + \frac{1}{1-2\nu}\frac{\partial}{\partial \theta^1}\left(\frac{\partial v_1}{\partial \theta^1} + \frac{\partial v_2}{\partial \theta^2} + \frac{1}{h}\frac{\partial v_3}{\partial x}\right) + \frac{2(1+\nu)}{E}F_1 = 0 \quad (1 \leftrightarrow 2) \tag{3.18}$$

$$\left(\Delta v_3 + \frac{1}{h^2}\frac{\partial^2 v_3}{\partial x^2}\right) + \frac{1}{1-2\nu}\frac{1}{h}\frac{\partial}{\partial x}\left(\frac{\partial v_1}{\partial \theta^1} + \frac{\partial v_2}{\partial \theta^2} + \frac{1}{h}\frac{\partial v_3}{\partial x}\right) + \frac{2(1+\nu)}{E}F_3 = 0$$

where Δ is the two-dimensional Laplace operator, and F_1, F_2, F_3 are components of the mass correction force which are known in the second stage. Since the resultant force and resultant moment of the mass correction forces equal zero in the second stage, we will have

$$F_1(\theta^1, \theta^2, x) = \sum_{n=2}^{\infty} F_1{}^{(n)}(\theta^1, \theta^2) P_n(x) \qquad (1 \leftrightarrow 2)$$

$$F_3(\theta^1, \theta^2, x) = \sum_{n=1}^{\infty} F_3{}^{(n)}(\theta^1, \theta^2) P_n(x) \tag{3.19}$$

Inserting the expansions (3.17) and (3.19) into (3.18), we obtain

$$\Delta v_{1,n}{}^2 + \frac{1}{1-2\nu}\frac{\partial}{\partial \theta^1}\left(\frac{\partial v_{1,n}{}^2}{\partial \theta^1} + \frac{\partial v_{2,n}{}^2}{\partial \theta^2}\right) = \tag{3.20}$$

$$= (4n^2 - 1)\left[\frac{1}{(1-2\nu)(2n+1)h}\frac{\partial v_{3,n-1}{}^2}{\partial \theta^1} - \frac{2(1+\nu)}{E}F_1{}^{(n-2)} + L_1{}^{(n-2)}\right] \qquad (1 \leftrightarrow 2)$$

$$\Delta v_{3,n-1}{}^2 = (2n-3)(2n-1)\left[-\frac{2(1+\nu)}{E}F_3{}^{(n-2)} + L_3{}^{(n-2)}\right] \qquad (n \geqslant 4)$$

Equations (3.20) form a system with the unknowns

$$v_{1,n}{}^2(\theta^1, \theta^2),\ v_{2,n}{}^2(\theta^1, \theta^2),\ v_{3,n-1}{}^2(\theta^1, \theta^2)$$

The functions $L_i{}^{(n-2)}(\theta^1, \theta^2)$ $(i = 1, 2, 3)$ are known since they depend on quantities determined in previous stages, in particular

$$L_i{}^{(2)}(\theta^1, \theta^2) = 0 \qquad (i = 1, 2, 3)$$

It should be noted that only the particular solution of the system (3.20) is of interest, and the construction of this solution presents no special difficulties since the first two equations correspond to the plane problem, and $v_{3,n-1}{}^2$ is determined from a Poisson-type equation.

BIBLIOGRAPHY

1. Green, A. E., On the linear theory of thin elastic shells. Proc. Roy. Soc. Vol. 266, №1325, 1962.

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